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B. S. Preetham , M. Anderson, and C. Richards

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Mathematical modeling of a four-stroke resonant engine for micro and mesoscale applications

B. S. Preetham,1 M. Anderson,2 and C. Richards1,a)
1School of Mechanical and Materials Engineering, Washington State University, P.O. Box 642920, Pullman, Washington 99164-2920, USA
2Department of Mechanical Engineering, University of Idaho, Moscow, Idaho 83844-0902, USA

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In order to mitigate frictional and leakage losses in small scale engines, a compliant engine design is proposed in which the piston in cylinder arrangement is replaced by a flexible cavity. A physics-based nonlinear lumped-parameter model is derived to predict the performance of a prototype engine. The model showed that the engine performance depends on input parameters, such as heat input, heat loss, and load on the engine. A sample simulation for a reference engine with octane fuel/air ratio of 0.043 resulted in an indicated thermal efficiency of 41.2%. For a fixed fuel/air ratio, higher output power is obtained for smaller loads and vice-versa. The heat loss from the engine and the work done on the engine during the intake stroke are found to decrease the indicated thermal efficiency. The ratio of friction work to indicated work in the prototype engine is about 8%, which is smaller in comparison to the traditional reciprocating engines. © 2014 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4903217]

I. INTRODUCTION

Over the past two decades, various engine configurations at small length scales have been studied that can produce power in the 1–1000 W range. Small scale engines can be categorized into microscale and mesoscale based on their physical dimensions. Engines whose dimensions are in the range of a few micrometers to millimeters are referred to as microscale engines, while those in the 1 mm to 10 cm size range are called mesoscale engines. Typically, both light and heavy hydrocarbon fuels are burned in these engines to obtain mechanical work or electrical energy. Most small scale engines are reciprocating engines with spark ignition and are modeled as an Otto cycle, whose theoretical efficiency depends on the compression ratio (CR) of the engine and the specific heat ratio of the working gas. In practice, performance is limited by heat, friction, throttling, and combustion losses. A scaling analysis on two-stroke piston engines revealed that a break-even in power occurs in an engine with dimensions 3 mm × 3 mm × 3 mm (or 27 mm³ size). This study indicates that it is possible to develop engines capable of producing power at micro and mesoscales, whose sizes are greater than 27 mm³. By addressing the bottlenecks, such as friction and leakage losses, it is possible to operate these engines more efficiently than previously reported.

In developing small scale engines, researchers have scaled down existing macro scale engines to small scales. In macro scale reciprocating engines, friction and heat losses are on the order of 10% and 30% of the input fuel chemical energy, respectively. Upon scaling down the engine size to micro and mesoscales, the heat transfer losses become dominant due to increased surface area to volume ratio and temperature gradients. For instance, a heat transfer study in a small-scale 125 cm³ two-stroke spark ignition engine showed that about 50% of input fuel chemical energy is lost due to heat transfer. To reduce heat loss from the combustion chamber, it has been suggested to insulate the combustor with a material having excellent insulating properties for both conduction and radiation.

Similarly at small scales, friction losses can become very dominant due to increased surface area to volume ratio and higher speeds. To reduce frictional losses at small scale, engines based on a free-piston engine are being developed. The free-piston design eliminates the crankshaft and bearing friction, thereby hypothetically improving the overall engine efficiency. The low frictional losses and pure linear motion of the piston nearly eliminate the need for lubrication. A four-stroke free-piston engine (engine bore = 62 mm and maximum stroke = 70 mm) when operated had a compression ratio of 6.5 and an expansion ratio (ER) of 10.5 resulting in an efficiency of 32%. An unrestricted expansion stroke resulted in obtaining additional work which was not possible in an Otto cycle based engine. Experimental results showed that the free-piston engine outperformed a traditional reciprocating engine in terms of efficiency with an identical geometric configuration. These features encouraged researchers to develop them at micro and mesoscales.

However, at small scales, the issues of poor sealing and leakage losses are exacerbated. In a traditional reciprocating engine, a typical tolerance value of piston-cylinder gap is about 10–20 μm, which is a very small fraction of cylinder bore. As length scales diminish, the ratio of piston-cylinder gap to cylinder bore becomes large, this results in increased leakage losses. The gravity of leakage loss is evident in the MEMS Internal Combustion (IC) engine. Suzuki et al. showed that about 18% of the total chamber mass was
This concept shares some features with free-piston engines, such as a variable compression ratio which allows for fuel flexibility. However, unlike a single piston free-piston engine which uses a rebound device to store energy, the flexible cavity itself performs the task of a rebound device thus enabling a reduction in engine size. There are also other advantages to the compliant engine design. Hydrocarbon emissions are reduced since the need for lubricant is removed and crevice volumes are, if not eliminated, substantially reduced. Also, the geometric complexity of the chamber is reduced significantly which leads to a reduction in cost. Finally, the variable compression ratio facilitates fuel flexibility as the compression ratio restricts the fuels that can be used in an engine to avoid knock. With the implementation of appropriate fuel injection/delivery systems, a wide variety of hydrocarbon fuels both liquid and gaseous could possibly be used.

In this article, a physics-based nonlinear mathematical model of the resonant engine operating on a four-stroke cycle is developed to predict the engine dynamics and engine performance. Typically, a reciprocating engine operation is modeled either as an open or closed cycle. In reciprocating engines with fixed stroke length, i.e., traditional IC engines, the piston dynamics during the intake and exhaust strokes are not very different, therefore they are sometimes modeled as a closed cycle operation (without intake and exhaust strokes). But for an engine with varying stroke lengths, this approach is debatable. In this study, the engine is modeled as an open cycle to accurately estimate the engine performance. The engine parameters used in the model correspond to a prototype engine and were experimentally determined. The performance is evaluated in terms of efficiency and output power. As a part of the study, the effects of heat transfer and work done on the engine during the intake stroke are examined. Also, the frictional work as a percent of total work output is calculated and compared with traditional reciprocating engines.
standard atmospheric pressure \( P_o \). The deformable corrugations of total stiffness \( x \) allow horizontal displacement \( x(t) \) of the right end of the volume, modeled to have a lump-parameter inertia \( m \). When the displacement of the mass \( m \) is zero, the volume \( V_o \) is \( V_o = LS \), where \( L \) is the nominal length and \( S \) is the cross-sectional area of the engine cavity or chamber. The movement of the mass \( m \) is impeded by the dampers \( b_f \) and \( b_p \). These dissipative elements model energy loss to friction and energy is converted to useful work, respectively. In addition, a damper with negative damping coefficient \( b_p (b_p < 0) \) is incorporated to model the work done on the engine during the intake stroke. For a steady state operation, the air temperature \( T_o + \Delta T(t) \), pressure \( P_o + \Delta P(t) \), density \( \rho_o + \Delta \rho(t) \), and engine volume \( V_o + \Delta V(t) \) undergo cyclic variation, where the ambient and time-varying cyclic components are denoted with the “\( o \)” subscript and \( \Delta \), respectively.

The nonlinear lumped parameter model is developed by applying conservation of mass, conservation of energy, and an ideal gas model to a moving control volume containing the air as working fluid. Newton’s second law for the piston or mass \( m \), and a mass-flow rate equation for working fluid flowing through the valve. Mathematical statements of these principles are

\[
\dot{M}_i - M_e = \frac{d}{dt} \left[ (\rho_o + \Delta \rho)(V_o + \Delta V) \right], \tag{1}
\]

\[
\frac{d}{dt} \left[ (\rho_o + \Delta \rho) c_v (V_o + \Delta V)(T_o + \Delta T) \right] + (P_o + \Delta P) \frac{d(V_o + \Delta V)}{dt} + h \Delta T = q(t) + \dot{M}_i e_i - M_e e_e, \tag{2}
\]

\[
\rho_o + \Delta \rho = \frac{P_o + \Delta P}{R(T_o + \Delta T)}, \tag{3}
\]

\[
\frac{m}{S^2} \Delta \dot{V} + \frac{(b_f + b_p)}{S^2} \Delta V + \frac{s}{S^2} \Delta V = \Delta P, \tag{4}
\]

\[
\dot{M}_{i,e} = -k \Delta P. \tag{5}
\]

In the set [Eqs. (1)–(5)], \( c_v \) is the constant volume heat capacity of air and is assumed constant over the temperature range, \( h \) is a coefficient that models conduction/convection heat losses from the moving control volume to the outside environment, \( k \) is the proportionality constant that relates the mass flow rate of working fluid and pressure drop across the valves, \( R \) is the mass-specific gas constant of air, \( e_i \) and \( e_e \) are the specific enthalpies of the air entering and exiting the engine, and \( M_i \) and \( M_e \) are the rates of mass of air entering and exiting the engine, respectively. The damping coefficient \( b_p \) is zero for cases where no external work is done on the engine during the intake process, and/or for other processes in the engine’s operating cycle.

If the heat added per cycle and initial conditions \( \Delta T(0) \), \( \Delta P(0) \), and \( \Delta V(0) \) are specified, Eqs. (1)–(5) constitute a nonlinear model for the determination of the intermediate thermodynamic state variables \( \Delta T(t) \), \( \Delta P(t) \), \( \Delta \rho(t) \), and \( \Delta V(t) \). As the heat added per cycle is periodic, it is assumed that the dependent variables \( \Delta T(t) \), \( \Delta P(t) \), \( \Delta \rho(t) \), and \( \Delta V(t) \) will also be periodic at steady state. Fig. 2 shows the heat-rate function \( q(t) \) for four-stroke operation. The heat addition pulse associated with combustion phenomenon is assumed impulsive for a duration \( t_q \). The heating pulse \( q(t) \) at a constant rate of \( q_{\text{HI}} \) is applied at the end of the compression stroke at \( t = t_1 \). When \( t_q \rightarrow 0 \), \( \lim_{t_q \rightarrow 0} q_{\text{HI}} t_q = E \), where \( E \) is the finite energy supplied to the engine per cycle during the heating pulse. The heat addition process is followed by expansion, exhaust, and intake processes. At the time \( t = t_1 + t_2 + t_3 + t_4 = t_f \), the cycle repeats, so that \( t_f \) is the cycle period of the engine. The time lapsed in each process need not be equal.

In the analysis, it is assumed that the heating pulse \( q_{\text{HI}} \) and the blowdown process (initial phase of exhaust process) occur at zero piston velocity for an infinitesimal duration, therefore \( V_3 = V_2 = 0 \) and \( V_4 = V_4 = 0 \). The assumption that the piston velocity is zero when the impulsive heating and blowdown are applied is a condition of resonance at steady state.

For subsequent analysis, a nondimensional scaling is adopted. The following scales are applied to the independent and dependent variables

\[
\Delta \dot{V} = \frac{\Delta V}{V_o}, \quad \Delta \dot{P} = \frac{\Delta P}{P_o}, \quad \Delta \dot{T} = \frac{\Delta T}{T_o}, \quad \Delta \dot{\rho} = \frac{\Delta \rho}{\rho_o}, \quad \bar{q} = \frac{q}{\rho_o V_o}, \quad t = t \omega. \tag{6a-f}
\]

where \( \omega = \sqrt{\frac{1}{s_h m}} \) is a reference frequency associated with the Helmholtz stiffness \( s_h = \rho_o c_v^2 S^2 V_o \) of the air within the engine and the overbar indicates a nondimensional independent or dependent variable. After application of these scales, the nonlinear model [Eqs. (1)–(5)] becomes

\[
\dot{M}_i - M_e = \rho_o c_v V_o \left[ \Delta \dot{\rho} (1 + \Delta V) + \Delta \dot{V} (1 + \Delta \rho) \right], \tag{7}
\]

\[
C \frac{d}{dt} \left[ (1 + \Delta \dot{\rho}) c_v (1 + \Delta V)(1 + \Delta \dot{T}) \right] + (1 + \Delta \dot{P}) \Delta \dot{V} + H \Delta \dot{T} = \Gamma \bar{q} + \frac{M_i e_i}{P_o V_o} - \frac{M_e e_e}{P_o V_o}, \tag{8}
\]

\[
M = \frac{P_o V_o (1 + \Delta \dot{P})(1 + \Delta \dot{V})}{RT_o (1 + \Delta \dot{T})}, \tag{9}
\]

\[
\Delta \dot{V} + 2 (\zeta + \zeta_p) \Delta \dot{V} + \Delta \dot{V} = \frac{1}{\gamma} \Delta \dot{P}, \tag{10}
\]

\[
\dot{M}_{i,e} = -k P_o \Delta \dot{P}. \tag{11}
\]

where

\[
C = \frac{\rho_o c_v T_o}{P_o}, \quad H = \frac{h T_o}{\rho_o P_o V_o}, \quad \Gamma = \frac{q_{\text{HI}}}{\rho_o P_o V_o}, \quad \zeta = \frac{b_f + b_p}{2 m_o}, \quad \zeta_p = \frac{b_p}{2 m_o}, \quad \Lambda = \frac{s}{s_h}. \tag{12a-f}
\]

In Eq. (9), \( M \) is the mass of working fluid in the flexible cavity. For analysis, the set [Eqs. (8)–(10)] is placed into state space format, the system model [Eqs. (8)–(10)] becomes
where $M$ variables and $T$ state takes the form shown in Fig. 3. The thermodynamic processes of impulsive heat addition (IGO), end of impulsive heat addition (IGC) and beginning of expansion stroke, the end of the expansion stroke and beginning of blowdown process (exhaust valve opens EO), end of blowdown process and beginning of displacement process, end of displacement process and exhaust valve closes (EC), and intake process begins with intake valve opening (IO), respectively. In the figure, CR is defined as $V_f/V_e$ and the ER as $V_e/V_f$. It is worth mentioning that the exhaust stroke 5→6 and the intake stroke 7→1 in the figure are idealized to be constant pressure processes. However, it can only happen if $\Delta P = 0$ given by Eq. (16).

In the heat addition process 2→3, the temperature inside the flexible cavity before and after the heat addition can be computed from Eqs. (14) and (15) and are given by

$$\Delta T_3 = \Delta T_2 + \frac{\Psi}{(1 + \Delta P_3)(1 + \Delta V_3)},$$

where

$$\Psi = \frac{q_{ht}}{\rho_v T_0 c_v T_0}.$$

The temperature inside the cavity at the end of blowdown process 4→5 can be computed by assuming an isentropic expansion from pressure $P_4$ to ambient pressure $P_5 = 1$ given by the following equation:

$$\Delta T_5 = \left(1 + \Delta T_4\right) \left(1 + \Delta P_4\right)^{\frac{1}{\kappa}} - 1,$$

where $\Delta P_4 = 0$.

The resulting volumes and temperatures $\Delta V_3, \Delta T_4, \Delta V_5, \Delta T_3$ from expansion and compression processes 3→4 and 1→2 were computed by numerically integrating [Eqs. (13)–(15)] from the initial conditions $\Delta V_3, \Delta T_3, V_3 = 0$ and $\Delta V_1, \Delta T_1, V_1 = 0$ with $M_{1,e} = 0$. On the other hand, the resulting volumes, temperatures, and pressures $\Delta V_6, \Delta T_6, \Delta P_6$ and $\Delta V_1, \Delta T_1, \Delta P_1$ from exhaust and intake strokes 5→6 and 7→1 were computed by numerically integrating [Eqs. (9), (11), and (13)–(16)] from the initial conditions $\Delta V_5, \Delta T_5, \Delta P_5, \Delta V_5 = 0$ and $\Delta V_7, \Delta T_7$. 

![FIG. 3. Four-stroke steady state P-V diagram.](image-url)

![FIG. 2. Heat-rate function $q(t)$ for four-stroke operation.](image-url)
$\Delta P_f$, $\Delta V_f = 0$. Since the state variables at 6 and 7 are assumed to be the same, therefore, $\Delta V_7 = \Delta V_6$, $\Delta P_f = \Delta P_6$, and $\Delta V_f = \Delta V_6$.

The system [Eqs. (9), (11), and (13)–(16)] can only reach steady state when the proper choice of load $b$ and initial conditions $\Delta V_1$, $\Delta P_1$, and $\Delta T$ are selected for cycle integration. Improper choice of load $b$ and initial conditions $\Delta V_1$, $\Delta P_1$, and $\Delta T$ would lead to monotonically drifting state variables $\Delta V$, $\Delta P$, and $\Delta T$ precluding periodic behavior. However, solving the system [Eqs. (9), (11), and (13)–(16)] iteratively will eventually result in a steady state solution. But, in order to obtain the solution quickly and accurately, we are adopting the below (function minimization) approach.

To determine the steady-state behavior of the engine $q_H$ was specified, initial load $b$ was selected, and integration was started from the quiescent initial state of $\Delta V_1 = \Delta V_t = \Delta P_1 = 0$. An example integration illustrating the first cycle in a transient dwell-up from this initial state is shown in Fig. 4. The volume $\Delta V_2$ and temperature $\Delta T_2$ were obtained by integrating the set [Eqs. (13)–(15)] from the initial conditions $\Delta V_2$, $\Delta T_2$, $\Delta V_1 = 0$ for a time $T_2$ such that $\Delta V_2 = 0$ with $M_{i,e} = 0$. Note that the time $T_2$ is not known “a priori,” and is determined during the integration. The volume $\Delta V_3$ and temperature $\Delta T_3$ were determined using Eqs. (18) and (19). The volume $\Delta V_4$ and temperature $\Delta T_4$ were obtained by integrating Eqs. (13)–(15) from the initial conditions $\Delta V_3$, $\Delta T_3$, $\Delta V_1 = 0$ for a time $T_3$ such that $\Delta V_4 = 0$ with $M_{i,e} = 0$. The time $T_4$ is not known “a priori,” and is determined during the integration. The temperature $\Delta T_5$ and volume $\Delta V_5$ were determined from Eqs. (21) and (22). Next, the volume $\Delta V_6$, temperature $\Delta T_6$, and pressure $\Delta P_6$, were obtained by integrating the set [Eqs. (9), (11), and (13)–(16)] from the initial conditions $\Delta V_6$, $\Delta T_6$, $\Delta P_6$, $\Delta V_1 = 0$ for a time $T_6$ such that $\Delta V_6 = 0$. Now, the state variables at 7 were obtained from $\Delta V_7 = \Delta V_6$, $\Delta T_7 = \Delta T_6$, and $\Delta P_7 = \Delta P_6$. Finally, a volume $\Delta V_7$, temperature $\Delta T_7$, and pressure $\Delta P_7$ were obtained by integrating the set [Eqs. (9), (11), and (13)–(16)] from the initial conditions $\Delta V_7$, $\Delta T_7$, $\Delta P_7$, $\Delta V_7 = 0$ until a time $T_7$ such that $\Delta V_7 = 0$. Like the time of compression $T_7$ and time of expansion $T_2$, the time of exhaust $T_2$, the time of intake $T_2$ are not known “a priori,” and are determined by the conditional integration. Since the system undergoes a transient phase, the engine operating cycle is not closed, i.e., $\Delta V_t \neq \Delta V_f$, $\Delta T_t \neq \Delta T_f$, and $\Delta P_t \neq \Delta P_f$. Therefore, to obtain a steady state solution, an objective function is defined that computes square error of state variables at 1 and 1′, and a function minimization computational procedure is adopted to determine proper choice of $\Delta V_1$, $\Delta T_1$, and $\Delta P_1$ for a specified heat input $q_H$ that would result in a steady-state periodic cycle. Once the objective function satisfies a specified tolerance the computation stops, the cycle closes (i.e., $\Delta V_t = \Delta V_f$, $\Delta T_t = \Delta T_f$, and $\Delta P_t = \Delta P_f$), and the engine begins to operate in steady state.

III. RESULTS AND DISCUSSION

The compliant resonant engine, shown schematically in Fig. 1, with $h = 0 \text{ W/K}$, $b_f = 1.5 \text{ N/s/m}$, $k = 6.8 \times 10^{-7} \text{ kg/s Pa} \pm 20\%$, $V_o = 21.8 \text{ cm}^3$ (2.08 cm in diameter and 1.6 cm of nominal length), $s = 3050 \text{ N/m}$, $b = 20 \text{ N/s/m}$, $b_{f,1} = -40 \text{ N/s/m}$, and $m = 0.18 \text{ kg}$, is treated as the reference engine. The numerical values of frictional damping coefficient $b_f$, mass-flow rate coefficient $k$, nominal volume $V_o$, stiffness $s$, and mass $m$ were experimentally determined for the prototype engine. The clearance volume $V_c$ is chosen to be 0.01% of the nominal volume $V_o$. It is defined as the minimum cavity volume of the engine and usually occurs at state 6. Standard temperature and pressure conditions of $T_o = 295 \text{ K}$ and $P_o = 93.4 \text{kPa}$ are chosen for the ambient state and for the thermodynamic properties of air. The properties of air used are $R = 286.9 \text{ J/kg K}$, $\rho = 1.09 \text{ kg/m}^3$, $c_v = 717.25 \text{ J/kg K}$, and $c = 346 \text{ m/s}$. The nonlinear model [Eqs. (7)–(11)] is used to predict the engine performance and behavior.

A. Operating cycle

A relationship between the flexible cavity pressure and volume for four-stroke operation is shown in Fig. 5. It is obtained using the nonlinear model [Eqs. (7)–(11)] at steady state operation by impulsively supplying heat $E = 176 \text{ J/cycle}$ at the end of the compression stroke 1→2 to the reference engine. The heat $E = 176 \text{ J/cycle}$ supplied corresponds to lean combustion (equivalence ratio $\Phi = 0.65$) of octane in the
engine. Equivalence ratio is defined as the ratio of actual fuel/air ratio to the stoichiometric ratio. It is a more informative parameter for defining the mixture composition.\textsuperscript{21} The figure shows that the pressure inside the flexible cavity at the end of the heat addition process (i.e., at $V_2 = 50.1\text{ cm}^3$) rises to $\sim 1.6\text{ MPa}$. Following the heat addition process $2 \rightarrow 3$, the engine expands adiabatically $3 \rightarrow 4$ to about ten times its nominal volume, which is $232.7\text{ cm}^3$. Next, the blowdown process $4 \rightarrow 5$ takes place resulting in an instantaneous drop in pressure to the ambient state $P_3 = 93.4\text{ kPa}$. Subsequently, the engine undergoes the displacement process $5 \rightarrow 6$ to the minimum volume $V_6 = 0.22\text{ cm}^3$, followed by the intake process $6 \rightarrow 7 \rightarrow 1$ until volume $V_7 = 87.8\text{ cm}^3$. Finally, an adiabatic compression $1 \rightarrow 2$ takes place resulting in completion of one operating cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 1$. It can be seen from the figure that the stroke lengths are unequal making the engine operation unique in comparison with traditional reciprocating engines. The nondimensional terms for this reference engine at $\Phi = 0.65$ are $C = 2.5$, $H = 0$, $\Gamma = 3488$, $\Lambda = 0.274$, and $\zeta = 0.24$.

To determine the work done per cycle for $\Phi = 0.65$, a numerical integration of the pressure-volume diagram (Fig. 5) for one operating cycle was performed and found to be 72.47 J. The corresponding indicated thermal efficiency is calculated to be 41.2\%. In addition, CR and ER are calculated and found to be 1.756 and 4.646, respectively.

### B. Equivalence ratio and heat input relationship

For a specific fuel, the equivalence ratio is dependent on the mass of fuel injected (or associated heat input) and the mass of air inducted into the engine. Fig. 6 graphically shows the relation between heat input and corresponding the equivalence ratio for various loads $b = 20, 25, \text{ and } 30\text{ N s/m}$. Each plot in the figure is obtained by supplying the engine with varying heat, and the corresponding equivalence ratios are calculated based on the mass of air inducted during the intake process $6 \rightarrow 7 \rightarrow 1$. It becomes clear from the figure that the relationship between the equivalence ratio and heat input is dependent on the load $b$ on the engine.

For a fixed load (say, $b = 20\text{ N s/m}$), the equivalence ratio holds a nonlinear relationship for smaller heat inputs and a linear relation for higher heat inputs. The point where the nonlinear relation changes to linear is defined as the inflection point (represented by an open circle symbol). Beyond the inflection point, the mass of air intake during intake process $6 \rightarrow 7 \rightarrow 1$ remains constant; as a result, the equivalence ratio is linearly proportional to heat input. The inflection point is indicative of the first occurrence of engine’s volume $V_b$, becoming equal to the clearance volume $V_c$. Such an occurrence results in an impulsive force on the piston at state 6 bringing it to abrupt stop. In a free piston engine, such impulsive forces occur when the piston hits the cylinder head at the top dead center (TDC). The sudden piston stoppage phenomenon can be treated as an elastic or an inelastic collision. For a conservative estimation of engine efficiency and performance, the collision at state 6 is assumed to be inelastic in nature. The fact that beyond the inflection point, the mass of air intake during process $6 \rightarrow 7 \rightarrow 1$ remains constant can be understood by analyzing the engine dynamics as discussed below.

In an inelastic collision, the piston or mass is assumed to lose all the kinetic energy irrevocably to the surroundings. As a result, the energy available with the spring-damper system (bellows-piston combination) after the collision is only the stored spring energy at state 6. Since there exists a lower bound on the engine volume (i.e., clearance volume $V_c$), the negative spring excursions are bounded $\Delta V_b = \Delta V_c$. Hence, the stored spring energy at state 6 is a fixed value beyond the inflection point. Beyond the inflection point, the intake stroke length remains constant, and so the mass of inducted air is also constant. Therefore beyond the inflection point, the equivalence ratio depends solely on the heat input in a linear relationship.

The change from a nonlinear to a linear relation is observed for loads $b = 20$ and $25\text{ N s/m}$. However, for load $b = 30\text{ N s/m}$, such a trend is not observed in the range $\Phi < 1.3$. It is hypothesized that transition occurs in the domain $\Phi > 1.3$.

A brief comparative study of elastic and inelastic collision modeling on the engine performance showed some interesting facts. The difference in indicated thermal efficiencies for $\Phi = 0.6$ is about 5\% between the elastic and inelastic cases, and it widens with equivalence ratio. For a fixed load (say, $b = 20\text{ N s/m}$), the resonant frequency in elastic collision is higher than the inelastic. A similar trend is observed for $b = 25\text{ N s/m}$. The effect of an inelastic collision at top dead center did not affect the engine performance significantly. Supporting plots and detailed discussions are included in the Appendix. These piston collisions (elastic or inelastic) can possibly destroy the engine. However, they can be avoided by timely adjusting the load and heat input to the engine.

### C. Effects of equivalence ratio and load on engine performance

Using the nonlinear model [Eqs. (7)–(11)], the effect of equivalence ratio $\Phi$ and load $b$ on the engine is determined in terms of engine efficiency, volume ratios (CR and ER), operating frequency, and output power.
1. Efficiency and volume ratios

For a fixed load \( b = 20 \text{ N/m} \), the effect of equivalence ratio on indicated thermal efficiency and volume ratios (CR and ER) for the reference engine are presented in Fig. 7. From the figure, it can be seen that efficiency increases with equivalence ratio. In the figure, the efficiency rises from 40.29% to 47.53% for an increase in equivalence ratio from \( \Phi = 0.6 \) to \( \Phi = 1.2 \). This indicates that it is desirable to operate the engine at higher equivalence ratios. Of course, in theory it cannot increase indefinitely due to the Carnot limit and in practice flammability limits impose the upper and lower bounds on equivalence ratio. Similar trends have been reported for a closed cycle modeling of a resonant heat engine.\(^{19}\) It has also been found that for a fixed load \( b = 20 \text{ N/m} \), an increase in indicated thermal efficiency from 40.29% to 47.53% results in an increase in output power from 870 W to 2100 W.

On the secondary axis of the figure, the corresponding compression and expansion ratios are plotted. For any given \( \Phi \), the expansion ratio is greater than the compression ratio, and the difference between them grows as \( \Phi \) increases. It is evident from the figure that the expansion ratio increases with \( \Phi \), while the compression ratio increases and remains unchanged for \( \Phi > 0.23 \). Therefore, it can be concluded that the indicated thermal efficiency is a stronger function of expansion ratio than of compression ratio. The point where \( \Phi = 0.23 \) is the inflection point for the reference engine. The unaltered value of CR beyond the inflection point (\( \Phi > 0.23 \)) can be understood by analyzing the engine dynamics during the processes 7\( \rightarrow \)1 and 1\( \rightarrow \)2. Since beyond the inflection point, the intake stroke length for process 7\( \rightarrow \)1 remains constant, the engine volume \( V_1 \) remains fixed. As the engine volume \( V_2 \) is purely dictated by the stored spring energy at state 1, its volume at state 2 is predetermined by volume \( V_1 \). So beyond the engine’s inflection point, since the volumes \( V_1 \) and \( V_2 \) are fixed, the compression ratio of the engine does not change. However, the expansion ratio increases with equivalence ratio (or heat input), as higher heat input gives rise to higher chamber pressure that pushes the piston further outward.

The effect of load on the indicated thermal efficiency and expansion ratio is shown in Fig. 8. Three loading conditions are shown \( b = 20, 25, \) and \( 30 \text{ N/m} \). It can be observed that for fixed \( \Phi \) (say, \( \Phi = 1.0 \)), as the load applied increases from \( 20 \text{ N/m} (\zeta = 0.240) \) to \( 25 \text{ N/m} (\zeta = 0.296) \), the efficiency drops monotonically from 45.75% to 43.45%. The efficiency drops further down to 28.34% as load increases to \( 30 \text{ N/m} (\zeta = 0.352) \). This reiterates that it is desirable to design an engine with as small damping \( \zeta \) as possible to operate efficiently.\(^{19,24}\) However, in practice, the minimum damping is limited by the physically achievable expansion ratios. On the secondary axis of the figure, corresponding expansion ratios for the three cases of \( 20 \text{ N/m}, 25 \text{ N/m}, \) and \( 30 \text{ N/m} \) are plotted. For any given \( \Phi \), the expansion ratio drops as the load applied increases. This is attributed to a decrease in the amplitude of piston oscillation. Similar such findings have been reported for a closed cycle modeling of a resonant heat engine.\(^{19}\) A practical volume ratio (ER) of 6 is obtained from the model for \( \Phi = 1.0 \) and \( b = 20 \text{ N/m} \).

2. Thermodynamic states

In order to explore the effects of equivalence ratio (or heat input) and load \( b \) on the thermodynamic states of an operating cycle, a P-V diagram is plotted. To capture the effect of equivalence ratio on the reference engine, P-V diagrams for various equivalence ratios are plotted in Figs. 9(a) and 9(b). Fig. 9(a) is plotted for equivalence ratios within the confines of inflection point, i.e., for \( \Phi = 0.12 \) and \( \Phi = 0.15 \), while Fig. 9(b) is plotted for equivalence ratios beyond the inflection point, i.e., for \( \Phi = 0.65 \) and \( \Phi = 1.0 \). These two figures together capture the effect of equivalence ratios on the thermodynamic states of the engine operating cycle 1\( \rightarrow \)2\( \rightarrow \)3\( \rightarrow \)4\( \rightarrow \)5\( \rightarrow \)6\( \rightarrow \)7\( \rightarrow \)1. From the figures, increasing the equivalence ratio \( \Phi \) results in higher cavity pressure and thereby an increase in amplitude of oscillation yielding higher expansion ratio. It can be seen from Fig. 9(a) that the intake and compression stroke lengths increase with \( \Phi \). However, in Fig. 9(b), i.e., for \( \Phi \) beyond the inflection point, the intake and compression stroke lengths remain constant for increasing \( \Phi \). This can be understood by analyzing the engine dynamics during the processes 7\( \rightarrow \)1 and 1\( \rightarrow \)2 as discussed in Sec. III C 1.
The effect of load on the engine is studied for a fixed heat input of 39.25 J/cycle as shown in Fig. 10. Upon increasing the load \( b \) from 20 N \( \text{s/m} \) to 30 N \( \text{s/m} \), the volume excursions decrease while the area enclosed by the P-V loop increases from 9.31 J/cycle to 12.1 J/cycle. The decreased volume excursions, especially the intake stroke length, results in reduced engine breathing capacity. Hence, less air gets ingested during the intake stroke. This results in an increase in the equivalence ratio from \( \Phi = 0.19 \) to \( \Phi = 1.09 \) since the heat input is fixed and ingested air is decreased. To maintain a fixed equivalence ratio for increasing load \( b \), the heat input should be accordingly reduced to tally with the reduced air intake.

3. Operating frequency

Fig. 11 shows a plot of engine operating frequency versus equivalence ratio for three loads of \( b = 20, 25, \) and 30 N \( \text{s/m} \). It is evident from the figure that the frequency of engine operation depends on both the load and the equivalence ratio. For a fixed load \( b = 20 \text{ N s/m} \), the resonant frequency decreases until \( \Phi = 0.23 \) (inflection point represented by an open circle symbol, point A). The decreasing frequency up to point A indicates that the nonlinearity caused by the thermo-physics of the working fluid is softening in nature.

Beyond the inflection point (point A), the operation frequency increases due to reduced travel distance of the piston arising from collision at state 6. That is, the engine cycle time is shorter or operating frequency is higher. A similar trend is observed for load \( b = 25 \text{ N s/m} \) and the point of inflection occurs at point B represented with an open circle symbol. Since for load \( b = 30 \text{ N s/m} \) there is no inflection point in the equivalence ratio range plotted, the behavior is different.

4. Output power

The engine output power is computed by only accounting for the energy dissipated from the useful damper \( b \) (also called load). Fig. 12 shows the output power and the operating frequency of the engine for loads from 12 N \( \text{s/m} \) to 30 N \( \text{s/m} \). The plots in the figure are generated for fixed equivalence ratios of \( \Phi = 0.6 \) and \( \Phi = 1.0 \). The plot shows that the stoichiometric fuel air mixture generates higher output power from the engine. The difference in output power from stoichiometric and lean combustion \( (\Phi) \) is larger at smaller loads compared to higher loads. It is worth mentioning that both the work output and the operating frequency determine the output power from the engine. From Fig. 12, it is clear that the difference in frequencies for \( \Phi = 0.6 \) and
The numerical values of useful work and operating frequency for $\Phi = 1.0$ at each loading condition are presented in Table I. It is apparent from the table that the decreasing output power with load is a result of a large drop in work output (or useful work) from the engine. Since the heat input and engine efficiency decrease for a fixed $\Phi$ and increasing load, their product, i.e., work output also follows the decreasing trend. The figure suggests that for a fixed $\Phi$, smaller loads yield higher output power and vice-versa. In other words, to maintain a fixed output power (say, 320 W) for increasing load ($b > 20$ N s/m), the fixed frequency and fixed speed conditions are found to be identical.

## 5. Fixed speed and fixed frequency

An engine’s mean piston speed is proportional to stroke length and operating frequency. In a traditional reciprocating engine, the piston speed and the frequency of crankshaft revolution (RPM) are related by a proportionality constant (i.e., fixed stroke length). Therefore, fixed speed conditions are the same as fixed frequency (RPM) conditions. However, in the resonant engine studied here, the stroke length varies with load and equivalence ratio. Therefore, in the resonant engine, a fixed speed operating condition may be different from a fixed frequency condition.

### TABLE I. Output power for various loading conditions for fixed equivalence ratio $\Phi = 1.0$.

<table>
<thead>
<tr>
<th>Load (N s/m)</th>
<th>Equivalence ratio</th>
<th>Heat input (J/cycle)</th>
<th>Operating frequency (Hz)</th>
<th>Useful work (J/cycle)</th>
<th>Output power (W)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>1.01</td>
<td>483</td>
<td>13.94</td>
<td>183</td>
<td>2548</td>
</tr>
<tr>
<td>15</td>
<td>1.03</td>
<td>389</td>
<td>13.95</td>
<td>155</td>
<td>2166</td>
</tr>
<tr>
<td>17</td>
<td>1.02</td>
<td>334</td>
<td>13.89</td>
<td>135</td>
<td>1882</td>
</tr>
<tr>
<td>20</td>
<td>1.04</td>
<td>283</td>
<td>13.70</td>
<td>117</td>
<td>1601</td>
</tr>
<tr>
<td>23</td>
<td>1.02</td>
<td>236</td>
<td>13.32</td>
<td>97</td>
<td>1293</td>
</tr>
<tr>
<td>25</td>
<td>1.01</td>
<td>212</td>
<td>12.76</td>
<td>87</td>
<td>1108</td>
</tr>
<tr>
<td>27</td>
<td>1.01</td>
<td>114</td>
<td>12.33</td>
<td>43</td>
<td>529</td>
</tr>
<tr>
<td>28</td>
<td>1.00</td>
<td>75</td>
<td>12.33</td>
<td>26</td>
<td>320</td>
</tr>
<tr>
<td>30</td>
<td>1.04</td>
<td>36</td>
<td>12.25</td>
<td>10</td>
<td>123</td>
</tr>
</tbody>
</table>

The effect of load on equivalence ratio for fixed speed and fixed frequency operations is plotted in Fig. 13. From the figure, for a fixed speed operation of 6 m/s, the engine demands a richer fuel-air mixture as the load increases. Similar behavior is observed for a fixed frequency operation of 13.6 Hz. Such trends are expected and show resemblance to traditional reciprocating engines. In a traditional engine, the fixed speed and fixed frequency conditions are identical as described above. However, in the resonant engine, the fixed speed and fixed frequency conditions are different; a feature unique to the prototype engine. This is very evident from the figure for higher loads ($b > 20$ N s/m). Interestingly for a load $b = 20$ N s/m, the fixed frequency and fixed speed conditions are found to be identical.

### D. Effect of intake work

In traditional reciprocating engines, work is done on the engine during an intake process. The effect of such intake work on the prototype engine performance is studied by incorporating a damper $b_{p1}$ to model the intake work. A negative sign of $b_{p1}$ implies work done on the engine, while a value $b_{p1} = 0$ N s/m indicates that no external work is done on the engine during an intake stroke. Therefore, for $b_{p1} = 0$, $-20$, and $-40$ N s/m at a fixed heat of 138.55 J/cycle, it is clear from the figure that increasing intake work results in decreasing expansion ratio, and so decreasing engine efficiency. This suggests that for a fixed heat input, the work done on the engine during the intake process unfavorably affects the indicated thermal efficiency. The corresponding numerical values are presented in Table II. It shows that for $b_{p1} = 0$, $-20$, and $-40$ N s/m, the corresponding indicated thermal efficiencies are 47.59%, 44.25%, and 38.41%. From the table, the increase in intake work from $b_{p1} = 0$ N s/m to $b_{p1} = -40$ N s/m results in the decrease in equivalence ratio from $\Phi = 1.15$ to $\Phi = 0.51$. This is because the heat input is fixed. Increasing equivalence ratio implies decreasing mass of intake air in to the engine.
E. Effect of heat loss

The heat loss coefficient $h$ models conduction/convection heat losses from the engine. Its effect on the reference engine performance is studied for a fixed heat input of 138.55 J/cycle. The study investigates for a wide range of $h$ and relevant results are presented in Fig. 15 and Table III. The heat loss coefficient $h$ relates to heat loss by a proportional relation. For instance, an increase in $h$ from 0.01 to 0.1 W/K results in a ten-fold rise in heat loss. The steady state results show that for a fixed heat input and varying heat loss coefficient $h$, the equivalence ratios are found to be nearly invariant. Results show that for a fixed heat input of 138.55 J/cycle and equivalence ratio of 0.5, the temperature $T_3$ at the end of heat addition process in each of the three cases $h=0.01, 0.1,$ and $3$ W/K is equal. From the figure, the P-V diagrams for cases $h=0.01$ and $0.1$ W/K are nearly identical, implying that the work output is nearly equal. The heat loss during the expansion process $3 \rightarrow 4$ is largest for $h=3$ W/K and is manifested as pressure drop in Fig. 15. Since lower cavity pressure results in smaller strokes, the engine efficiency for $h=3$ W/K is found to be the lowest.

The numerical values of efficiency and useful work are presented in Table III.

In the above Sections III D and III E, the engine performance has been studied by independently observing the effects of friction and heat loss. Here, its performance is studied in the presence of both friction and heat loss. The performance is estimated by a nondimensional friction work defined as a ratio of the friction work to indicated work. In traditional macroscale reciprocating engines, this nondimensional quantity varies between 10% and 100% and depends on the load on the engine. During the idling condition, i.e., at zero load, the nondimensional friction work is about 100%, and it drops to 10% at full load. Thus for intermediate loads, its value varies between the limits 10% and 100%. At small length scales, the lower limit rises well above 10% due to the increase in friction and decrease in indicated thermal efficiency with engine size. This implies that for the resonant engine, the nondimensional friction work of 10% can be considered to be satisfactory. For this study, the nonlinear model [Eqs. (7)–(11)] is used with $h=0.1$ W/K, $b_p=0$ N s/m, $b_f=1.5$ N s/m, $b_f=20$ N s/m, $k=6.8 \times 10^{-1} \text{kg/s Pa} \pm 20\%$, $V_o=21.8 \text{cm}^3$ (2.08 cm in diameter and 1.6 cm of nominal length), $s=3050 \text{N/m}$, and $m=0.18 \text{kg}$ in a lean condition $\Phi=0.5$. The results showed that the nondimensional friction work in the engine is about 8%. Further analysis showed that for a fixed equivalence ratio $\Phi=0.5$, upon increasing the load on the engine from 20 N s/m to 29 N s/m, the nondimensional friction dropped from 8% to 5% (Fig. 16). This implies that the resonant engine’s ability to utilize indicated work increases with higher loads. This study reiterates that the friction loss in the resonant engine with respect to indicated work is less than the traditional reciprocating engines.

The numerical values of efficiency and useful work are presented in Table III.

\begin{table}[h]
\centering
\caption{Effect of intake work on reference engine performance.}
\begin{tabular}{|c|c|c|c|}
\hline
Intake work load, $b_p$ (N s/m) & 0 & $-20$ & $-40$ \\
Heat input (J/cycle) & 138.55 & 138.55 & 138.55 \\
Equivalence ratio & 1.74 & 1.15 & 0.51 \\
Expansion ratio & 8.37 & 6.43 & 4.13 \\
Compression ratio & 1.05 & 1.22 & 1.76 \\
Indicated thermal efficiency (%) & 47.59 & 44.25 & 38.41 \\
Useful work (J/cycle) & 60.34 & 56.02 & 47.99 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{Effect of heat loss on reference engine performance.}
\begin{tabular}{|c|c|c|c|}
\hline
Heat loss coefficient $h$ (W/K) & 0.01 & 0.1 & 3 \\
Heat input (J/cycle) & 138.55 & 138.55 & 138.55 \\
Equivalence ratio & 0.505 & 0.504 & 0.504 \\
Heat lost (J/cycle) & 0.377 & 3.625 & 63.55 \\
Indicated thermal efficiency (%) & 37.6 & 37.4 & 30.8 \\
Useful work (J/cycle) & 47.95 & 47.63 & 39.14 \\
\hline
\end{tabular}
\end{table}
at same length scale. Therefore, a compliant resonant engine has the potential to outperform the contemporary engines at small scale. This encourages us to pursue further developmental studies to materialize the concept.

Though the engine presented here is four-stroke operated, it would be interesting to consider the possibility of a two-stroke operation. However, there are some foreseeable challenges, such as timely delivery of fuel/air mixture and efficient scavenging. Interestingly, a literature review on two-stroke free-piston engines revealed that current efforts are directed towards developing engines based on two-stroke opposed-piston opposed-cylinder designs.25,26

IV. CONCLUSIONS

In this article, work towards the development of a resonant engine operating on a four-stroke Otto cycle principle at small length scales has been carried out. To mitigate friction and leakage losses at small scale, a compliant engine design was proposed in which the piston-cylinder assembly was replaced by a flexible cavity. The aim of the study has been to demonstrate the potential of the compliant engine at small scale. This has been accomplished by developing a mathematical model that can predict the performance of a prototype engine.

The mathematical model was derived by applying conservation of mass, conservation of energy, an ideal gas model to a moving control volume containing the air as working fluid, Newton’s second law for the piston, and mass-flow rate equation for working fluid flowing through the valve. The model was solved to predict the performance of a prototype engine for various loading conditions over a range of equivalence ratios. The steady state results showed that the engine performance depends on input parameters, such as heat input, heat loss, and load on the engine. For a fixed load, as the heat input to the engine was increased, a sudden change in engine behavior was observed. The point where sudden change occurred was defined as inflection point. It has been found that the occurrence of inflection point also depends on the engine load. The cause for occurrence of inflection point was traced to an impulsive force on the piston at state 6. Such a phenomenon has been modeled to be both elastic and inelastic collision. The efficiency results showed that the elastic and inelastic models did not differ significantly. However, for a conservative estimation, the inelastic model was chosen for further analysis. The results showed that engine’s indicated thermal efficiency was found to be a strong function of expansion ratio. It has been observed that the heat loss and work done on the engine during the intake stroke unfavorably affected the indicated thermal efficiency. A case study numerical simulation showed that the ratio of friction work to indicated work in the prototype engine can be brought down to 5%, which is smaller than friction loss in traditional reciprocating engines at that length scale. This result reinforces that a compliant resonant engine has the potential to outperform the contemporary engines at small scale and can be one of the plausible solutions to mitigate frictional and leakage losses at that length scale.

APPENDIX: ELASTIC COLLISION

To have an optimistic estimation of engine efficiency and performance, the collision at TDC (state 6, Fig. 3) is assumed to be elastic in nature. In the elastic collision, the piston or mass is assumed to restore all the energy, and the energy available with the spring-mass damper system (bellows-piston combination) is higher than in the inelastic collision. The elastic collision is modeled by assuming that the piston engages with a stiff spring $(s_{stiff} = 600 \times s)$ as it approaches TDC state 6. The stiff spring models the kinetic energy that would have been lost if the collision was inelastic. The recovered kinetic energy is utilized in subsequent process, i.e., the intake stroke 7→1.

In this section, a brief comparative study of elastic and inelastic collision modeling on the engine performance is presented.

Fig. 17 shows a plot of indicated thermal efficiency for the reference engine for varying equivalence ratio. As pointed out in the main article for equivalence ratios within the confines of the inflection point (i.e., $\Phi < 0.23$), the engine...
volume $V_0$ is above the clearance volume $V_c$ and no collision occurs. Therefore for $\Phi < 0.23$, there is not distinction between the elastic and inelastic cases in Fig. 17. But for equivalence ratios beyond the inflection point (i.e., $\Phi > 0.23$), since the collision occurs, the engine efficiency is higher for elastic collision than for the inelastic. The difference in indicated thermal efficiencies for $U = 0.6$ is about 5% between the elastic and inelastic cases, and it widens with equivalence ratio.

A plot of engine cycle frequency for two loads $b = 20$ and 25 N s/m for varying equivalence ratio is shown in Fig. 18. The plots in the figure are similar to Fig. 11 in the main article. For $\Phi < 0.23$, there is no distinction in frequency for elastic and inelastic collision for load $b = 20$ N s/m. However, for $\Phi > 0.23$, a significant difference in elastic and inelastic collisions can be observed. For a fixed load (say, $b = 20$ N s/m), the resonant frequency in elastic collision is higher than the inelastic. This can be explained on the lines of a simple oscillator. A simple oscillator’s natural frequency reduces in the presence of a damper given by the equation $\omega_d = \omega_n \sqrt{1 - \zeta^2}$, where $\omega_n$ is the natural frequency of the oscillator, $\omega_d$ is the damped natural frequency of the oscillator, and $\zeta$ is the damping ratio. By analogy, the damping effect caused by the inelastic collision will reduce the engine’s operating frequency. A similar trend is observed for the load $b = 25$ N s/m.